§17.9. Virasoro operators

(We shall extend the above construction to the case of a reductive f.d. Lie algebra $\mathfrak{g}$. It will be more convenient to use here this notation instead of $\mathring{\mathfrak{g}}$))

[· A Lie algebra is $\boxed{\text{reductive}}$ if its adjoint representation is completely reducible. More concretely, a Lie algebra is reductive if it is a direct sum of a semisimple Lie algebra and an abelian Lie algebra: $\mathfrak{g} = \mathfrak{s} + A$
(背占)]

· We have the decomposition of $\mathfrak{g}$ into a direct sum of ideals

$$(17.9.1) \qquad \mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)} \oplus \cdots$$

where $\mathfrak{g}_{(0)}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}_{(i)}$ with $i \geq 1$ are simple.

· We fix on $\mathfrak{g}$ a non-degenerate invariant bilinear form $(\cdot|\cdot)$ so that (17.9.1) is an orthogonal decomposition.
可对角化 ⇒ ∃非退化不变双线性型 且 Affine 和 finite 是可对角化.
(since $[\mathfrak{g}_{(i)}, \mathfrak{g}_{(i)}] = \mathfrak{g}_{(i)}$ ($\mathfrak{g}$) the by $([x,y]|z) = (x|[y,z])$ and $[\mathfrak{g}_{(i)}, \mathfrak{g}_{(j)}] = 0$ if $i \neq j$.)

·(We shall assume that the restriction of $(\cdot|\cdot)$ to each $\mathfrak{g}_{(i)}$ with $i \geq 1$ is the normalized invariant form / Assume $(e_i^j | e_i^j) = 1$, where e_i^j, $j \in I$ is the basis of $\mathfrak{g}_{(i)}$, $i \geq 1$. and that $\mathfrak{g}_{(0)}$ and the form $(\cdot|\cdot)$ restricted to it are (as described in Rem 12.8) positive definite bilinear form).

· Define a normalized invariant form on $\mathfrak{g}$. i.e. $(\alpha|\alpha) = 2$ if $\alpha \in \Delta_\ell$.
$(\cdot|\cdot)|_{\mathfrak{g}_{(i)}}$, $i \geq 1$ is normalized invariant form. $(\cdot|\cdot)|_{\mathfrak{g}_{(0)}}$ is positive definite bilinear form.

(We call such form a normalized invariant form on $\mathfrak{g}$.)

· We let:

$$\tilde{\mathcal{L}}(\mathfrak{g}) = \bigoplus_{i \geq 0} \tilde{\mathcal{L}}(\mathfrak{g}_{(i)}), \text{ where } \tilde{\mathcal{L}}(\mathfrak{g}_{(i)}) = \mathcal{L}(\mathfrak{g}_{(i)}) + \mathbb{C}K_i.$$

($\mathfrak{g}_{(i)}$ 作为子李代数之中心)

also let (c.f. §7.4)

$$\hat{\mathcal{L}}(\mathfrak{g}) = \tilde{\mathcal{L}}(\mathfrak{g}) + \mathbb{C}d, \text{ where } d|_{\tilde{\mathcal{L}}(\mathfrak{g}_{(i)})} = \underline{\#}\, t\frac{d}{dt}, \quad d(K_i) = 0.$$

· The Lie algebra $\tilde{\mathcal{L}}(\mathfrak{g})$ and $\hat{\mathcal{L}}(\mathfrak{g})$ are called <u>affine algebras associated to the reductive Lie alg $\mathfrak{g}$</u>.

The subalgebra $\tilde{\mathcal{L}}(\mathfrak{g}_{(i)})$ (resp. $\tilde{\mathcal{L}}(\mathfrak{g}_{(i)}) + \mathbb{C}d$)) are called $\boxed{\text{components}}$ of $\tilde{\mathcal{L}}(\mathfrak{g})$ (resp. $\hat{\mathcal{L}}(\mathfrak{g})$)

· Note that. $\mathfrak{c} = \mathfrak{g}_{(0)} + \sum_{i \geq 1} \mathbb{C}K_i$ is the center of $\tilde{\mathcal{L}}(\mathfrak{g})$ and $\hat{\mathcal{L}}(\mathfrak{g})$.

As before, we identify $\mathfrak{g}$ with the subalgebra of $1 \otimes \mathfrak{g}$. Let $\bar{\mathfrak{H}}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{g} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{H}} \oplus \bar{\mathfrak{n}}_+$ be a triangular decomposition of $\mathfrak{g}$.

The subalgebra $\mathfrak{H} = \bar{\mathfrak{H}} + \mathfrak{c} + \mathbb{C}d$ is called the Cartan subalgebra of $\hat{\mathcal{L}}(\mathfrak{g})$.

The triangular decomposition: $\hat{\mathcal{L}}(\mathfrak{g}) = \mathfrak{n}_- \oplus \mathfrak{H} \oplus \mathfrak{n}_+$ is defined in the same way as in §7.6. (P102).

$$\mathfrak{n}_- = (t^{-1}\mathbb{C}[t^{-1}] \otimes (\bar{\mathfrak{n}}_+ + \bar{\mathfrak{H}})) + \mathbb{C}[t^{-1}] \otimes \bar{\mathfrak{n}}_-$$

$$\mathfrak{n}_+ = (t\,\mathbb{C}[t] \otimes (\bar{\mathfrak{n}}_- + \bar{\mathfrak{H}})) + \mathbb{C}[t] \otimes \bar{\mathfrak{n}}_+$$

?

, since $\Delta_+^{re} = \{\alpha \in \Delta^{re}, \text{ with } n>0\} \cup \mathring{\Delta}_+$

· For $\Lambda \in \mathfrak{H}^*$, (we denote (as before) its restriction to $\bar{\mathfrak{H}}$ by $\bar{\Lambda}$.) denote $\Lambda|_{\bar{\mathfrak{H}}} := \bar{\Lambda}$

As before, define $\delta \in \mathfrak{H}^*$ by : $\delta|_{\bar{\mathfrak{H}}+\mathbb{C}} = 0$, $\langle \delta, d \rangle = 1$

· Given $\Lambda \in \mathfrak{H}^*$, we denote (as before) by $L(\Lambda)$ the irreducible $\hat{\mathcal{L}}(\mathfrak{g})$-module which admits a non-zero vector v_Λ such that : $n_+(v_\Lambda) = 0$ and $h(v_\Lambda) = \langle \Lambda, h \rangle v_\Lambda$ for $h \in \mathfrak{H}$.

Using uniqueness of $L(\Lambda)$, we clearly have:

怎么理解这个张量积？

$\Lambda_{(i)} = \Lambda|_{\mathfrak{H} \cap \hat{\mathcal{L}}(\mathfrak{g}_{(i)})}$ (12.9.2) $L(\Lambda) = \bigotimes_{i \ge 0} L(\Lambda_{(i)})$ (张量积？) 为什么不是直和？

(where $\Lambda_{(i)}$ denote the restriction of Λ to $\mathfrak{H}_{(i)} := \mathfrak{H} \cap \hat{\mathcal{L}}(\mathfrak{g}_{(i)})$) and $L(\Lambda_{(i)})$ is the $\hat{\mathcal{L}}(\mathfrak{g}_{(i)})$-module with highest weight $\Lambda_{(i)}$.

· We let k_i, the eigenvalue of K_i on $L(\Lambda)$, be the i-th level of Λ, and let $k = (k_0, k_1, \cdots)$. i.e. $k_i = \langle \Lambda, K_i \rangle$

Define $c(k) = \sum_i c(k_i)$, $h_\Lambda = \sum_i h_{\Lambda(i)}$, $m_\Lambda = \sum_i m_{\Lambda(i)}$

⌈ $c(k) = \frac{k(\dim \mathfrak{g})}{k + h^\vee}$, $h_\Lambda = \frac{(\Lambda+2\rho|\Lambda)}{2(k+h^\vee)}$ if $V = L(\Lambda)$.

(conformal anomaly) (vacuum anomaly)

$m_\Lambda = \frac{|\Lambda+\rho|^2}{2(k+h^\vee)} - \frac{|\rho|^2}{2h^\vee}$ (modular anomaly) ⌋

Due to (12.9.2), $\underbrace{ch_{L(\Lambda)} = \prod_i ch_{L(\Lambda_{(i)})}}$ and

(12.9.3) $\chi_\Lambda := e^{-m_\Lambda \delta} ch_{L(\Lambda)} = \prod_i \chi_{\Lambda(i)}$ $\quad ch_{V_1 \oplus V_2} = ch_{V_1} + ch_{V_2}$

⌈ Prob. normalized character $\chi_\Lambda = e^{-m_\Lambda \delta} ch_{L(\Lambda)}$ ⌋

· Let V be a restricted $\hat{\mathcal{L}}(\mathfrak{g})$-module s.t. K_i acts as $k_i I$ and $k_i \neq -h_i^\vee$ (where $h_i^\vee$ is the dual Coxeter number of $\hat{\mathcal{L}}(\mathfrak{g}_{(i)})$.

Let $T_n^{(i)}$ be the Sugawara operators for $\underset{\sim}{\hat{\mathcal{L}}(\mathfrak{g}_{(i)})}$, and let ($n \in \mathbb{Z}$) $\hat{\mathcal{L}}(\mathfrak{g}_{(i)}) = \tilde{\mathcal{L}}(\mathfrak{g}_{(i)}) + \mathbb{C}d_i$

(12.9.4) $L_n^{(i)} = \frac{T_n^{(i)}}{2(k_i+h_i^\vee)}$, $L_n^{\mathfrak{g}} = \sum_i L_n^{(i)}$

The operators $L_n^{\mathfrak{g}}$ are called the $\boxed{\text{Virasoro operators}}$ for the $\mathfrak{g}$-module V.

Then letting $d_n \longmapsto L_n^{\mathfrak{g}}$, $c \longmapsto c(k)$ extends V to a module over $\hat{\mathcal{L}}(\mathfrak{g}) + Vir$

· Note also the following useful formula (cf. (12.8.5)).

(12.9.5) $L_0^{\mathfrak{g}} = \sum_i \frac{\Omega_i}{2(k_i+h_i^\vee)} - d$

where Ω_i is the Casimir operator for $\hat{\mathcal{L}}(\mathfrak{g}_{(i)})$.

⌈ $L_0^{(i)} = \frac{T_0^{(i)}}{2(k_i+h_i^\vee)}$, $L_0^{\mathfrak{g}} = \sum_i \frac{T_0^{(i)}}{2(k_i+h_i^\vee)} - d$

from (12.8.5) $T_0 = -2(K+h^\vee)d + \Omega$

$d = d_0 + d_1 + \cdots$

⌋

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§12.10. Coset Vir-module.

In the remainder of this chapter, we let $\mathfrak{g}$ be a reductive f-d. Lie algebra with a normalized invariant form $(\cdot|\cdot)$ and let $\dot{\mathfrak{g}}$ be a reductive subalgebra of $\mathfrak{g}$ such that $(\cdot|\cdot)|_{\dot{\mathfrak{g}}}$ is non-degenerate.

Let $\mathfrak{g} = \bigoplus_{i\geq 0} \mathfrak{g}_{(i)}$ and $\dot{\mathfrak{g}} = \bigoplus_{i\geq 0} \dot{\mathfrak{g}}_{(i)}$ be the decompositions (12.9.1) of $\mathfrak{g}$ and $\dot{\mathfrak{g}}$. Let $(\cdot|\cdot)^*$ be a normalized invariant form on $\dot{\mathfrak{g}}$, which coincides with $(\cdot|\cdot)$ on $\dot{\mathfrak{g}}_{(0)}$. Due to uniqueness of the invariant bilinear form on simple Lie algebra. we have for $x, y \in \dot{\mathfrak{g}}_{(s)}$, $s\geq 1$

$$(x_{(r)} \mid y_{(r)}) = j_{sr}(x|y)^*$$

if $x, y \in \dot{\mathfrak{g}}_{(1)}$ then by definition $(x|y)^* = (x|y) = (x_{(1)} | y_{(1)})$

where $x_{(r)}$ denotes the projection of x on $\mathfrak{g}_{(r)}$ and j_{sr} is a (positive) number independent of x and y; we let $j_{0r} = 1$.

$x \in \dot{\mathfrak{g}}_{(s)} \Rightarrow x \in \mathfrak{g}$ i.e. $x = x_{(0)} + x_{(1)} + \dots$ where $x_{(i)} \in \mathfrak{g}_{(i)}$

The numbers j_{sr} $(s, r \geq 0)$ are called $\boxed{\text{Dynkin indices}}$

- The inclusion homomorphism $\varphi : \dot{\mathfrak{g}} \to \mathfrak{g}$ induces in an obvious way the inclusion hom: $\mathcal{L}(\dot{\mathfrak{g}}) \to \mathcal{L}(\mathfrak{g})$. This lifts uniquely to a homomorphism $\tilde{\varphi} : \tilde{\mathcal{L}}(\dot{\mathfrak{g}}) \to \tilde{\mathcal{L}}(\mathfrak{g})$ by letting $\tilde{\varphi}(\dot{K}_s) = \sum_r j_{sr} K_r$, which extends to a homomorphis $\hat{\varphi} : \hat{\mathcal{L}}(\dot{\mathfrak{g}}) \to \hat{\mathcal{L}}(\mathfrak{g})$ by letting $\hat{\varphi}(d) = d$.

(Here and further the overdot refers to an object associated to $\dot{\mathfrak{g}}$).

- Let V be a restricted $\tilde{\mathcal{L}}(\mathfrak{g})$-module s.t. K_i acts as $k_i I$, $k_i \neq -h_i^\vee$. Via $\tilde{\varphi}$, this is a $\tilde{\mathcal{L}}(\dot{\mathfrak{g}})$-module with $\dot{K}_i$ acting as $\dot{k}_i I$, where:

$\forall v \in V,\ \mathfrak{g}_\alpha(v) = 0$ for all but a finite number of positive roots α

$$(12.10.1) \qquad \dot{k}_s = \sum_i j_{si} k_i \quad \text{by } \tilde{\varphi}(\dot{K}_s) = \sum_r j_{sr} K_r$$

We shall assume that $\dot{k}_i \neq -\dot{h}_i^\vee$, let (see (12.9.4))

$$L_n^{\mathfrak{g},\dot{\mathfrak{g}}} = L_n^{\mathfrak{g}} - L_n^{\dot{\mathfrak{g}}}$$

prop 12.10.

a) The operators $L_n^{\mathfrak{g},\dot{\mathfrak{g}}}$ commute with $\tilde{\mathcal{L}}(\dot{\mathfrak{g}})$.

proof: by Lem 12.8 one has: $[x^{(m)}, T_n] = 2(K + h^\vee) m x^{(m+n)}$

then $\forall \dot{x}^{(m)} \in \tilde{\mathcal{L}}(\dot{\mathfrak{g}})$, $\dot{x}^{(m)} = \sum_i t_i \dot{x}_{(i)}^{(m)}$ where $\dot{x}_{(i)}^{(m)} \in \tilde{\mathcal{L}}(\dot{\mathfrak{g}}_{(i)})$

hence $[\dot{x}^{(m)}, L_n^{\mathfrak{g}}] = [\sum_i t_i \dot{x}_{(i)}^{(m)}, \sum_i L_n^{(i)}] = \sum_i t_i [\dot{x}_{(i)}^{(m)}, L_n^{(i)}]$

$$= \sum_i t_i \frac{[\dot{x}_{(i)}^{(m)}, T_n^{(i)}]}{2(k_i + h_i^\vee)} = \sum_i t_i m \dot{x}_{(i)}^{(m+n)} = m\dot{x}^{(m+n)}$$

similarly: $[\dot{x}^{(m)}, L_n^{\dot{\mathfrak{g}}}] = [\sum_i t_i \dot{x}_i^{(m)}, \sum_i \dot{L}_n^{(i)}] = \sum_i t_i [\dot{x}_{(i)}^{(m)}, \dot{L}_n^{(i)}]$

$$= \sum_i t_i \frac{[\dot{x}_{(i)}^{(m)}, \dot{T}_n^{(i)}]}{2(\dot{k}_i + \dot{h}_i^\vee)} = \sum_i t_i m \dot{x}_{(i)}^{(m+n)} = m\dot{x}^{(m+n)}$$

thus $[L_n^{\mathfrak{g},\dot{\mathfrak{g}}}, \dot{x}^{(m)}] = 0$. #

b) The map $d_n \mapsto L_n^{g,\dot{g}}$, $c \mapsto c(k) - \dot{c}(\dot{k})$ defines a representation of Vir on V. (V is a restricted $\hat{L}(g)$-module)

proof: we need to prove $\rho: Vir \to gl(V)$ via: $d_n \mapsto L_n^{g,\dot{g}}$, $c \mapsto c(k) - \dot{c}(\dot{k})$ is a Lie algebra homomorphism.

Explicitly, this is to say ρ should be a linear map and it should satisfy $\rho([x,y]) = [\rho(x), \rho(y)]$ for all $x, y \in Vir$.

- linear map:
- $\rho([d_i, d_j]) = \rho((i-j) d_{i+j} + \frac{1}{12}(i^3 - i)\delta_{i,-j} c)$

$= (i-j) L_{i+j}^{g,\dot{g}} + \frac{1}{12}(i^3-i)\delta_{i,-j}(c(k) - \dot{c}(\dot{k}))$

on the other hand.

$[\rho(d_i), \rho(d_j)] = [L_i^{g,\dot{g}}, L_j^{g,\dot{g}}] = [L_i^{g,\dot{g}}, L_j^{\dot{g}}]$ (since $L_j^{\dot{g}} \in V_c(\hat{L}(\dot{g}))$)

$= [L_i^g, L_j^{\dot{g}}] - [L_i^{\dot{g}}, L_j^{\dot{g}}] = [L_i^g, L_j^g] - [L_i^{\dot{g}}, L_i^{g,\dot{g}} + L_i^{\dot{g}}]$

$= [L_i^g, L_j^g] - [L_i^{\dot{g}}, L_j^{\dot{g}}]$

since $[L_i^g, L_j^g] = [\sum_q L_i^{(q)}, \sum_q L_j^{(q)}] = \sum_q \frac{[T_i^{(q)}, T_j^{(q)}]}{(2(k_q + h_q^\vee))^2}$

by (12.8.8) $= \sum_q \frac{1}{2(k_q + h_q^\vee)} ((i-j)T_{i+j}^{(q)} + \delta_{i,-j}\frac{1}{6}(i^3 - i)(\dim g_{(q)}) k_q)$

$= (i-j) L_{i+j}^g + \delta_{i,-j} \frac{1}{6} \cdot \frac{1}{2}(i^3 - i) \sum_q \dim g_{(q)} k_q \cdot \frac{1}{k_q + h_q^\vee}$

$= (i-j) L_{i+j}^g + \delta_{i,-j} \frac{1}{12}(i^3-i) c(k)$

[Recall: (12.8.10) $c(k) = \frac{k(\dim g)}{k + h^\vee}$]

Similarly: $[L_i^{\dot{g}}, L_j^{\dot{g}}] = (i-j) L_{i+j}^{\dot{g}} + \delta_{i,-j}\frac{1}{12}(i^3 - i)\dot{c}(\dot{k})$

Then $[\rho(d_i), \rho(d_j)] = (i-j) L_{i+j}^{g,\dot{g}} + \frac{1}{12}(i^3-i)\delta_{i,-j}(c(k) - \dot{c}(\dot{k}))$

$= \rho([d_i, d_j])$.

- $\rho([d_i, c]) = \rho(0) = 0 = [\rho(d_i), \rho(c)] = [L_i^{g,\dot{g}}, c(k) - \dot{c}(\dot{k})]$

Directly to prove: let $d_n . v := \rho(d_n).v$, $c.v := \rho(c).v$

① $(a d_n + b d_m).v = \rho(a d_n + b d_m).v \underset{\text{由}\rho\text{的线性}}{=} a\rho(d_n).v + b\rho(d_m).v$

② $d_n.(av + bw) = \rho(d_n).(av+bw) = L_n^{g,\dot{g}}.(av+bw)$

③ $[d_n, d_m].v = \rho([d_n, d_m]).v = \cdots$

where $a, b \in F$ (field), $v, w \in V$.

#

· The Vir - module defined by prop 12.10 is called the $\boxed{\text{coset Vir - module.}}$

· Choose Cartan subalgebras $\bar{H}$ and $\dot{\bar{H}}$ of g and $\dot{g}$ such that $\dot{\bar{H}} \subset \bar{H}$. Choose a triangular decomposition $g = \bar{n}_- \oplus \bar{H} \oplus \bar{n}_+$; then we have the induced triangular decomposition $\dot{g} = \dot{\bar{n}}_- \oplus \dot{\bar{H}} \oplus \dot{\bar{n}}_+$, where $\dot{\bar{n}}_\pm = \bar{n}_\pm \cap \dot{g}$. We have the associated triangular decompositions: $\hat{\mathcal{L}}(g) = n_- \oplus H \oplus n_+$, $\hat{\mathcal{L}}(\dot{g}) = \dot{n}_- \oplus \dot{H} \oplus \dot{n}_+$ etc. and we have: $\psi(\dot{H}) \subset H$, $\psi(\dot{n}_+) \subset n_+$, etc.

· Let $P_+ = \{\lambda \in H^* \mid \lambda|_{H \cap \hat{\mathcal{L}}(g_{(i)})} \in P_{+(i)}$ for $i \geq 1$ and $\lambda|_{H \cap g_{(0)R}}$ is real and $\langle \lambda, K_0 \rangle > 0\}$ $P_{(i)} = \{\lambda \in (H \cap \hat{\mathcal{L}}(g_{(i)}))^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}\}$

加了修改?

be the set of $\boxed{\text{dominant integral weights}}$ $P_{+(i)} = \{\lambda \in P_{(i)} \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0\}$ for $\hat{\mathcal{L}}(g)$.

Let $P_+^k = \{\lambda \in P_+ \mid \lambda(K_i) = k_i \ (i = 0, 1, \cdots)\}$.

· Fix $\lambda \in P_+^k$.

Remark: (1) $\hat{\mathcal{L}}(g)$-module $L(\Lambda)$ is unitarizable.

[By (12.9.2) $L(\Lambda) = \bigotimes_{i \geq 0} L(\Lambda_{(i)})$

· Th. 11.7 b): Every integrable highest-weight module $L(\Lambda)$ over $g(A)$ is unitarizable.

· §11.12 ?

┘

(2) Viewed as a $\hat{\mathcal{L}}(\dot{g})$-module, then we have $L(\Lambda) = \bigoplus_{\lambda \in \dot{P}_+^k} \dot{L}(\lambda)$, where $\dot{L}(\lambda)$ is a $\hat{\mathcal{L}}(\dot{g})$-modules.

[Recall: P.198 Prop 11.8: Let $A \subset g(A)$ be an ω_0-invariant subalgebra which is normalized by an element $h \in \operatorname{Int} X_c$ (i.e. $[h, A] \subset A$). Then with respect to A, the module $L(\Lambda)$ ($\Lambda \in P_+$) decomposes into an orthogonal (with respect to $H(\,)$) direct sum of irreducible h-invariant submodules.

. It is easy to see $\hat{\mathcal{L}}(\dot{g})$ is ω_0-invariant and $[h, \hat{\mathcal{L}}(\dot{g})] \subset \hat{\mathcal{L}}(\dot{g})$ for some $h \in \operatorname{Int} X_c$. Actually $h = d$.

┘

(3) We denote the multiplicity of occurrence of $\dot{L}(\lambda)$ in (2) by $\operatorname{mult}_\Lambda(\lambda; \dot{g})$, and $|\operatorname{mult}_\Lambda(\lambda, \dot{g})| < +\infty$.

[· Since the eigenspaces of d on $L(\Lambda)$ are finite-dimensional

· $d \in \operatorname{Int} X_c$, since $(\alpha_i^\vee \mid d) = 0$ $(i = 1, \ldots, l)$, $(\alpha_0^\vee \mid d) = a_0$, and $\operatorname{Int} X = \{h \in H_R \mid \langle \alpha, h \rangle < 0$ only for a finite number of $\alpha \in \Delta_+\}$ P44

· By P.197 Lem 11.8.

┘

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